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Dear Paul

This is to confirm that on May 15, 1987 the journal *Linear Algebra and its Applications* received the paper

Leonard Pairs and Dual Polynomial Sequences by Paul Terwilliger

for possible publication. The paper was reviewed favorably but a revision was never submitted.

Sincerely,

Richard A. Brualdi
Editor-in-Chief, *Linear Algebra and its Applications*

LEONARD PAIRS AND

DUAL POLYNOMIAL SEQUENCES

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Running head: Leonard pairs

Proofs to be sent to :

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LEONARD PAIRS AND DUAL SEQUENCES: Symbols used.

ϵ Greek epsilon

{} curley brackets

* star

Σ capital Greek sigma

$\mathbb{R}$ bold face R

<,> brackets

[,] brackets

$\rightarrow, \leftrightarrow$ arrow

θ Greek theta

δ Greek delta

σ Greek sigma

λ Greek lambda

β Greek beta

γ Greek gamma

ω Greek omega

$\oplus$ circle with bars

$\cup$ union symbol

$\subset$ inclusion symbol

$\Box$ box

$\mathbb{I}$ bold face I.

Note to Printer: No subscripts are starred.

$V_{i+1}{}^{*}$ should read V^{*}_{i+1}

ABSTRACT

Let V be a finite dimensional vector space over $\mathbb{R}$. We call diagonalizable linear transformations $A, A^* \in \mathrm{End}_{\mathbb{R}}(V)$ a Leonard pair with respect to orderings $V_0, V_1, \ldots, V_d$ and $V_0^*, V_1^*, \ldots, V_e^*$ of their respective maximal eigenspaces, if

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq e) \quad (V_{-1}^* = V_{e+1}^* = 0),$$

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d) \quad (V_{-1} = V_{d+1} = 0),$$

$$E_iA^*E_j \neq 0 \quad \text{if} \quad |i-j| = 1 \qquad (0 \leq i,j \leq d), \text{ and}$$

$$E_i^*AE_j^* \neq 0 \quad \text{if} \quad |i-j| = 1 \qquad (0 \leq i,j \leq e),$$

where $E_i : V \to V_i$ $(0 \leq i \leq d)$, $E_i^* : V \to V_i^*$ $(0 \leq i \leq e)$ are the projections. The pair is thin if $\dim(V_i) = \dim(V_j^*) = 1$ $(0 \leq i \leq d,\ 0 \leq j \leq e)$. In this paper we classify the thin Leonard pairs by obtaining a natural 1-1 correspondence between them and the pairs of finite 3-term recurrent polynomial sequences that are dual in the sense of Leonard's theorem. The polynomials are essentially the eigenbases for the transformations. We then define an (infinite dimensional) Leonard algebra over $\mathbb{R}$ by generators and relations, and show all thin Leonard pairs arise from irreducible representations of these algebras. A Leonard pair A, A^* is quasi A-bipartite (resp. quasi A*-bipartite) if $E_i^*AE_i^* = 0$ $(1 \leq i \leq e-1)$ (resp. if $E_iA^*E_i = 0$ $(1 \leq i \leq d-1)$), and symmetric if A and A^* are each self-adjoint with respect to the same positive definite bilinear form on V. We then extend the methods used on the thin case to obtain a classification of all symmetric quasi A- or A*- bipartite Leonard pairs.

1. INTRODUCTION.

Let V be a finite dimensional vector space over $\mathbb{R}$, and let $A \in \mathrm{End}_{\mathbb{R}}(V)$ be any diagonalizable linear transformation. Let $V_0, V_1, \ldots, V_d$ be the maximal eigenspaces for A, and let $E_i: V \to V_i$ $(0 \leq i \leq d)$ be the corresponding projections (so $E_0 + E_1 + \ldots + E_d = \mathbb{I}$, $E_iE_j = \delta_{ij}E_i$, $0 \leq i, j \leq d$). Let $A^* \in \mathrm{End}_{\mathbb{R}}(V)$ also be diagonalizable, with eigenspaces V_i^* and projections E_i^* $(0 \leq i \leq e)$. Throughout this paper, and unless otherwise indicated, $x_{-1} = x_{d+1} = x_{e+1}^* = 0$ for any variable x.

DEFINITION 1.1 The above transformations A, A^* form a <u>Leonard pair</u> (with respect to the given orderings of their eigenspaces) if

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq e) \qquad (1.1)$$

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d). \qquad (1.2)$$

We also assume the nondegeneracy conditions

$$E_iA^*E_j \neq 0 \quad \text{if} \quad |i-j| = 1 \qquad (0 \leq i, j \leq d) \qquad (1.3)$$

$$E_i^*AE_j^* \neq 0 \quad \text{if} \quad |i-j| = 1 \qquad (0 \leq i, j \leq e). \qquad (1.4)$$

We call d, e the <u>diameters</u> of the pair. $\mathrm{Spec}(A) \in \mathbb{R}^{d+1}$ and $\mathrm{Spec}(A^*) \in \mathbb{R}^{e+1}$ will denote the lists of distinct eigenvalues of A and A^*, respectively, with the induced ordering.

Leonard pairs $A, A^* \in \mathrm{End}_{\mathbb{R}}(V)$ and $C, C^* \in \mathrm{End}_{\mathbb{R}}(W)$ are <u>equivalent</u> if $\mathrm{Spec}(A) = \mathrm{Spec}(C)$, $\mathrm{Spec}(A^*) = \mathrm{Spec}(C^*)$, with $A = \sigma^{-1} C \sigma$, $A^* = \sigma^{-1} C^* \sigma$ for some invertible $\mathbb{R}$- linear map $\sigma: V \to W$. We do not distinguish between equivalent pairs.

DEFINITION 1.2. The Leonard pair A, A^* in Definition 1.1 is <u>thin</u> if $\dim(V_i) = \dim(V_j^*) = 1$ $(0 \le i \le d, 0 \le j \le e)$ (so $d = e = \dim(V) - 1$). A, A^* is <u>irreducible</u> if V contains no nonzero, proper A- and A^*- invariant subspace. A, A^* is <u>quasi</u> A- <u>bipartite</u> (resp. <u>quasi</u> A^*- <u>bipartite</u>) if $E_i^* A E_i^* = 0$ $(1 \le i \le e-1)$ (resp. $E_i A^* E_i = 0$ $(1 \le i \le d-1)$). A, A^* is <u>symmetric</u> if there exists a positive definite bilinear form $\langle\ ,\ \rangle$ on V with respect to which A and A^* are self-adjoint (i.e. for $C = A$ or A^*, $\langle Cu,v\rangle = \langle u,Cv\rangle$ for all $u, v \in V$).

In this paper we begin a classification of Leonard pairs. Our main motivation is their appearance in the study of P- and Q- polynomial association schemes (indeed, the matrices A, x_E in Terwilliger [6,Thm. 3] form a symmetric Leonard pair). We shall develop this connection in a future paper. Here we focus on the thin case and the symmetric quasi-bipartite case, obtaining the following results.

(1) We show the classification of thin Leonard pairs is equivalent to the finite dimensional version of Leonard's Theorem (Bannai and Ito[3, p263]), which goes as follows. Call a finite polynomial sequence $(u(x))_d := \{u_0(x)=1, u_1(x),\dots,u_d(x), u_{d+1}(x)\}$ 3-<u>term</u> <u>recurrent</u> if

$$xu_i(x) = c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x) \qquad (0 \le i \le d-1)$$

and

$$xu_d(x) = c_d u_{d-1}(x) + a_d u_d(x) + u_{d+1}(x)$$

for some $c_i, a_i, b_i \in \mathbb{R}$ $(0 \leq i \leq d)$, where $c_i \neq 0$ $(1 \leq i \leq d)$, $b_i \neq 0$ $(0 \leq i \leq d-1)$, and $c_0, b_d = 0$. The c_i, a_i, b_i are the <u>parameters</u> of $(u(x))_d$. A vector $\underline{\theta} = (\theta_0, \theta_1, \ldots, \theta_d)$ $(\theta_i \in \mathbb{R},\ 0 \leq i \leq d)$ is an <u>eigenvalue sequence</u> of $(u(x))_d$ if the θ_i are distinct roots of $u_{d+1}(x)$. Then 3-term recurrent polynomial sequences $(u(x))_d$ and $(u^*(x))_d$ are <u>dual</u> with respect to eigenvalue sequences $\underline{\theta}$ and $\underline{\theta}^*$, respectively, if

$$u_i(\theta_j) = u_j^*(\theta_i^*) \qquad (0 \leq i, j \leq d). \tag{1.5}$$

Leonard's Theorem provides a classification of all dual sequences.. By that theorem the $(u(x))_d$ and $(u^*(x))_d$ are essentially <u>Askey-Wilson polynomials</u>[1,2], including limiting cases. We establish the above mentioned equivalence by obtaining a natural 1-1 correspondence between the thin Leonard pairs and dual sequences. The polynomials are essentially eigenbases for the transformations. This correspondence gives a classification of the the thin Leonard pairs.

(2) If $A, A^* \in \mathrm{End}_{\mathbb{R}}(V)$ is a thin Leonard pair, then there exists a nondegenerate symmetric bilinear form on V with respect to which A and A^* are self-adjoint. The form is unique up to scalar multiplication. A, A^* is symmetric if and only if the corresponding dual polynomial sequences are each orthogonal.

(3) The <u>Leonard algebra</u> L <u>on</u> a, a^* <u>with parameters</u> $\beta, \gamma, \gamma^*, \delta, \delta^*, \omega \in \mathbb{R}$ is defined (over $\mathbb{R}$) by generators a, a^* and relations

$$a^2a^* - \beta aa^*a + a^*a^2 + \gamma(a^*a + aa^*) + \delta a^* + \gamma^* a^2 + \omega a \in \mathrm{Center}(L)$$

$$a^{*2}a - \beta a^*aa^* + aa^{*2} + \gamma^*(aa^* + a^*a) + \delta^* a + \gamma a^{*2} + \omega a^* \in \mathrm{Center}(L),$$

where $\mathrm{Center}(L) = \{b \mid b \in L,\ ba=ab \text{ and } ba^*=a^*b\}$.

We say a finite dimensional representation $\sigma: L \to \mathrm{End}_{\mathbb{R}}(V)$ is self-adjoint (resp. symmetric) if there exists a nondegenerate symmetric bilinear form (resp. positive definite form) $\langle\ ,\ \rangle$ on V with respect to which $\sigma(a)$ and $\sigma(a^*)$ are self-adjoint. Then any thin Leonard pair is of the form $\sigma(a), \sigma(a^*)$, with σ an irreducible self-adjoint representation of a Leonard algebra on a, a^*. The algebra is unique if the diameter $d \geq 3$. Our converse is slightly weaker. Assume the parameter β of a Leonard algebra L is such that q $(q+q^{-1}=\beta)$ is not a primitive nth root of unity for any n $(n \geq 3)$. Then $\sigma(a), \sigma(a^*)$ is a thin Leonard pair for any finite dimensional, irreducible, symmetric representation σ of L.

We then consider quasi-bipartite Leonard pairs and prove

(4) Let the Leonard pair A, A^* be irreducible, symmetric, and quasi A- or A^*- bipartite. Then A, A^* is thin. This leads to a classification (thm. 4.2) of all symmetric quasi A- or A^*- bipartite Leonard pairs.

We finish with some open problems.

2. THIN LEONARD PAIRS AND DUAL SEQUENCES

In this section we obtain a natural 1 - 1 correspondence between the thin Leonard pairs and dual sequences. We begin with a few technicalities.

LEMMA 2.1. Let $A, A^* \in \mathrm{End}_{\mathbb{R}}(V)$ be a thin Leonard pair. Then (i) V has no proper A- (resp. A^*-) invariant subspaces containing V_0^* (resp. V_0). In particular, (ii) any $f \in \mathrm{End}_{\mathbb{R}}(V)$ satisfying $fV_0^*=0$ and $fA=Af$ (resp. $fV_0=0$ and $fA^*=A^*f$) is 0.

Proof. Statement (i) is immediate from (1.3),(1.4). Statement (ii) is obtained by applying (i) to $\{v \mid v \in V, fv = 0\}$. □

DEFINITION 2.2. The valencies $k_0=1, k_1,\dots,k_d$ of a 3-term recurrent sequence $(u(x))_d$ are given by

$$k_i = b_0 b_1 \dots b_{i-1} / c_1 c_2 \dots c_i \qquad (0 \leq i \leq d), \tag{2.1}$$

where the c_j, b_j are parameters of $(u(x))_d$.

THEOREM 2.3. (i) Let 3-term recurrent sequences $(u(x))_d$ and $(u^*(x))_d$ be dual with respect to eigenvalue sequences $\underline{\theta} = (\theta_0, \theta_1, \dots, \theta_d)$ and $\underline{\theta}^* = (\theta_0^*, \theta_1^*, \dots, \theta_d^*)$. Then the transformations $A, A^* \in \mathrm{End}_{\mathbb{R}}(V)$, $V = \mathbb{R}[x]/(u_{d+1}(x))$ defined by

$$Au_i(x) = xu_i(x), \qquad A^*u_i(x) = \theta_i^* u_i(x) \qquad (0 \leq i \leq d), \tag{2.2}$$

form a thin Leonard pair with $\mathrm{Spec}(A) = \underline{\theta}$, $\mathrm{Spec}(A^*) = \underline{\theta}^*$.

Furthermore (ii) any thin Leonard pair may be uniquely realized in this way.

Proof of (i). Let c_i, a_i, b_i, k_i and $c_i^*, a_i^*, b_i^*, k^*_i$ $(0 \leq i \leq d)$ be the parameters and valencies of $(u(x))_d$ and $(u^*(x))_d$, respectively, and set

$$u_j^* = \sum_{i=0}^{d} u_i(\theta_j) k_i u_i(x) \qquad (0 \leq j \leq d).$$

We must verify (1.1) - (1.4). But (1.1) holds by (2.2) and

$$Au_i(x) = c_i u_{i-1}(x) + a_i u_i(x) + b_i u_{i+1}(x) \qquad (0 \leq i \leq d). \tag{2.3}$$

Now (1.2) holds, for by (2.3) we have

$$Au_j^* = \theta_j u_j^* \qquad (0 \leq j \leq d), \tag{2.4}$$

and using (1.5) we obtain

$$A^* u_j^* = c_j^* u_{j-1}^* + a_j^* u_j^* + b_j^* u_{j+1}^* \qquad (0 \leq j \leq d). \tag{2.5}$$

Line (1.3),(1.4) follows from $b_i, b_i^* \neq 0$ $(0 \leq i \leq d-1)$, $c_i, c_i^* \neq 0$ $(1 \leq i \leq d)$, so A, A^* is the desired Leonard pair.

<u>Proof of (ii)</u>. Now let $A, A^* \in \mathrm{End}_{\mathbb{R}}(W)$ be any thin Leonard pair, with $\mathrm{Spec}(A) = (\theta_0, \theta_1, \ldots, \theta_d)$ and $\mathrm{Spec}(A^*) = (\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$.

The <u>Standard</u> <u>A</u>-<u>basis</u> of the pair is a sequence $X = \{v_0, v_1, \ldots, v_d\}$ $(0 \neq v_i \in V_i^*, 0 \leq i \leq d)$ where $v_0 + v_1 + \ldots + v_d \in V_0$. The <u>Standard</u> <u>A</u>*-<u>basis</u> is $X^* = \{v_0^*, v_1^*, \ldots, v_d^*\}$ $(0 \neq v_i^* \in V_i, 0 \leq i \leq d)$, where $v_0^* + v_1^* + \ldots + v_d^* \in V_0^*$. X and X^* are unique up to scalar multiplication.

We note the standard bases exist, for pick any nonzero $v^* \in V_0$. Then $X = \{E_0^* v^*, E_1^* v^*, \ldots, E_d^* v^*\}$ must span V by (i) of Lemma 2.1, and is therefore a standard basis. The construction of X^* is similar.

(As an example, in the proof of (i) the vectors $v_i = k_i u_i(x)$, $v_i^* = k_i^* u_i^*$ $(0 \leq i \leq d)$ form standard A- and A*- bases, respectively).

By (1.1) - (1.3), we now have

$$Av_i = b_{i-1}v_{i-1} + a_i v_i + c_{i+1}v_{i+1} \qquad (v_i \in X,\ 0 \leq i \leq d) \qquad (2.6)$$

$$A^*v_i^* = b_{i-1}^* v_{i-1}^* + a_i^* v_i^* + c_{i+1}^* v_{i+1}^* \qquad (v_i^* \in X^*,\ 0 \leq i \leq d) \qquad (2.7)$$

for some $c_i, a_i, b_i, c_i^*, a_i^*, b_i^* \in \mathbb{R}$, $(0 \leq i \leq d)$, where $c_i, c_i^* \neq 0$ $(1 \leq i \leq d)$, $b_i, b_i^* \neq 0$ $(0 \leq i \leq d-1)$, and $c_0, c_0^*, b_d, b_d^* = 0$. Now set

$$k_i = b_0 b_1 \ldots b_{i-1} / c_1 c_2 \ldots c_i \qquad (0 \leq i \leq d)$$

and

$$k_i^* = b_0^* b_1^* \ldots b_{i-1}^* / c_1^* c_2^* \ldots c_i^* \qquad (0 \leq i \leq d).$$

Define the nondegenerate symmetric <u>bilinear form</u> $\langle , \rangle$ <u>of</u> A, A* on W by

$$\langle v_i, v_j \rangle = \delta_{ij} k_i \qquad (v_i, v_j \in X,\ 0 \leq i,j \leq d). \qquad (2.8)$$

Then one verifies A and A* are self-adjoint with respect to $\langle , \rangle$, and that up to scalar multiplication, $\langle , \rangle$ is unique with this property.

In particular, there exists a $c \in \mathbb{R}$ where

$$\langle v_i^*, v_j^* \rangle = c\delta_{ij} k_i^* \qquad (v_i^*, v_j^* \in X^*,\ 0 \leq i,j \leq d).$$

We now define the <u>dual A-basis</u> $Y = \{u_0, u_1, \ldots, u_d\}$ and <u>dual A*-basis</u> $Y^* = \{u_0^*, u_1^*, \ldots, u_d^*\}$ by

$$u_i = v_i / k_i \qquad u_i^* = v_i^* / k_i^* \qquad (0 \leq i \leq d).$$

Then Y and Y* are, up to scalar multiplication, the dual bases for X and

X^*, respectively, with respect to $\langle , \rangle$. Now let $u_i(x)$, $u_i^*(x)$ $(0 \le i \le d+1)$ be the unique polynomials satisfying

$$u_i(A)u_0 = u_i, \qquad u_i^*(A^*)u_0^* = u_i^* \qquad\qquad (0 \le i \le d+1).$$

Then $(u(x))_d$ and $(u^*(x))_d$ are 3-term recurrent with parameters the c_i, a_i, b_i and c_i^*, a_i^*, b_i^* $(0 \le i \le d)$ from (2.6), (2.7). To see they are dual with respect to Spec(A), Spec(A*), we compute $\langle u_i, u_j^* \rangle$ $(0 \le i,j \le d)$ two ways. First,

$$\begin{aligned}\langle u_i, u_j^* \rangle &= \langle u_i(A)u_0, u_j^* \rangle \\ &= h \sum_{m=0}^{d} \langle u_i(\theta_m)v_m^*, u_j^* \rangle \qquad (\text{where } h^{-1}u_0 = v_0^* + v_1^* + \ldots + v_d^*) \\ &= chu_i(\theta_j) \qquad (\text{where } \langle v_i^*, u_j^* \rangle = c\delta_{ij}) \\ &= \langle u_0, u_0^* \rangle u_i(\theta_j)\end{aligned}$$

and by symmetry

$$\langle u_i, u_j^* \rangle = \langle u_0, u_0^* \rangle u_j^*(\theta_i^*) \qquad\qquad (0 \le i,j \le d).$$

Now $\langle u_0, u_0^* \rangle \ne 0$ by the nondegeneracy of $\langle , \rangle$, so (1.5) holds. Finally, we note the maps between the sets of thin Lenard pairs and dual sequences implicit above and in (i) are inverses, establishing (ii). □

We may use the bilinear form (2.8) of a thin Leonard pair to interpret the orthogonality of the associated polynomials, as follows.

THEOREM 2.4. Let $(u(x))_d$ and $(u^*(x))_d$ be dual sequences, with associated thin Leonard pair A, A^*. Then the following are equivalent.

(i) The bilinear form of A, A^* is positive definite

(ii) $(u(x))_d$ is an orthogonal sequence

(iii) $(u^*(x))_d$ is an orthogonal sequence.

Proof. By Bannai and Ito[3, p276], $(u(x))_d$ is an orthogonal sequence if and only if its parameters satisfy $b_i c_{i+1} > 0$ $(0 \leq i \leq d-1)$. By (2.8), this is equivalent to the bilinear form of A, A^* being positive definite, so (i)$\leftrightarrow$(ii). (i)$\leftrightarrow$(iii) is similar. □

3. THE LEONARD ALGEBRA

In this section we show any thin Leonard pair is naturally associated with an irreducible representation of an certain infinite dimensional algebra over $\mathbb{R}$.

DEFINITION 3.1. The Leonard algebra L on a, a^* with parameters $\beta, \gamma, \gamma^*, \delta, \delta^*, \omega \in \mathbb{R}$ is defined (over $\mathbb{R}$) by generators a, a^* and relations

$$a^2a^* + a^*a^2 - \beta aa^*a + \gamma(a^*a+aa^*) + \delta a^* + \gamma^*a^2 + \omega a \in \text{Center}(L) \quad (3.1)$$

$$a^{*2}a + aa^{*2} - \beta a^*aa^* + \gamma^*(aa^*+a^*a) + \delta^*a + \gamma a^{*2} + \omega a^* \in \text{Center}(L). \quad (3.2)$$

Note 1. If the parameter β above satisfies $\beta = q + q^{-1}$ with $q \in \mathbb{R}$, then L is conveniently described by generators a, a^*, e, f, f^* and relations

$$aa^* - qa^*a = qe$$

$$ea - qae = \gamma(a^*a + aa^*) + \delta a^* + \gamma^* a^2 + \omega a + f$$

$$a^*e - qea^* = \gamma^*(aa^*+a^*a) + \delta^* a + \gamma a^{*2} + \omega a^* + f^*$$

$$f, f^* \in \text{Center}(L).$$

<u>Note 2</u>. A Leonard algebra on a, a^* with parameters $\beta, \gamma, \gamma^*, \delta, \delta^*, \omega$ is also a Leonard algebra on $a' = ha + k$, $a^{*\prime} = h^*a^* + k^*$ $(h, h^*, k, k^* \in \mathbb{R},\ h, h^* \neq 0)$, with parameters β, $h\gamma + k(\beta-2)$, $h^*\gamma^* + k^*(\beta-2)$, $h^2\delta - 2hk\gamma - (\beta-2)k^2$, $h^{*2}\delta^* - 2h^*k^*\gamma^* - (\beta-2)k^{*2}$, $hh^*\omega - 2hk^*\gamma - 2h^*k\gamma^* - 2(\beta-2)kk^*$. Therefore, after substituting (and possibly interchanging a, a^*) we may assume $(\beta, \gamma, \gamma^*, \delta, \delta^*, \omega)$ is of the form $(\beta, 0, 0, 1 \text{ or } -1, 1 \text{ or } -1, \omega)$ (with $\omega \geq 0$), $(\beta, 0, 0, 0, 1 \text{ or } -1, 1 \text{ or } 0)$, $(\beta, 0, 0, 0, 0, 1 \text{ or } 0)$, $(2, 1, 1, 0, 0, \omega)$, or $(2, 1, 0, 0, 1 \text{ or } 0 \text{ or } -1, 0)$. Leonard pairs are classified into "types" in Bannai and Ito[3, p263], but those types depend on the specific representations as well as the forms presented here.

We will need the following technical fact.

LEMMA 3.2. Let $r_0, r_1, \ldots, r_n$ $(n \geq 3)$ be real numbers, with $r_0, r_1, \ldots, r_{n-1}$ distinct and $r_0 = r_n$. Suppose

$$r_{i+3} - (\beta+1)r_{i+2} + (\beta+1)r_{i+1} - r_i = 0 \qquad (0 \leq i \leq n-3) \qquad (3.3)$$

for some $\beta \in \mathbb{R}$. Then the $q \in \mathbb{C}$ with $q + q^{-1} = \beta$ is a primitive nth root of unity.

Proof. There exist constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ where

(a) $r_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i}$ $(0 \leq i \leq n)$ if $\beta \neq 2$ or -2

(b) $r_i = \alpha_1 + \alpha_2 i + \alpha_3 i^2$ $(0 \leq i \leq n)$ if $\beta = 2$

(c) $r_i = \alpha_1 + (-1)^i(\alpha_2 + \alpha_3 i)$ $(0 \leq i \leq n)$ if $\beta = -2$.

In case (a), the distinctness of $r_0, r_1, \ldots, r_{n-1}$ implies $q^i \neq 1$ $(3 \leq i \leq n-1)$, and $r_0 = r_n$ implies either $q^n = 1$ or $\alpha_3 = q^n \alpha_2$. But the second possibility implies $r_1 = r_{n-1}$, so q must be a primitive nth root of unity, as desired. Cases (b) and (c) are similar. □

DEFINITION 3.3. A finite dimensional representation (over $\mathbb{R}$) of a Leonard algebra L is a homomorphism $\sigma: L \to \mathrm{End}_{\mathbb{R}}(V)$, with V a finite dimensional vector space over $\mathbb{R}$. The representation is irreducible if V has no proper, nonzero $\sigma(L)$- invariant subspaces. The representation is self-adjoint (resp. symmetric) if there exists a nondegenerate symmetric bilinear form (resp. positive definite form) on V with respect to which $\sigma(a)$ and $\sigma(a^*)$ are self-adjoint. We now have

THEOREM 3.4.

(i) Any thin Leonard pair is of the form $\sigma(a), \sigma(a^*)$, with σ an irreducible, self-adjoint representation of a Leonard algebra on a, a^*. The algebra is unique if the diameter $d \geq 3$. Conversely,

(ii) suppose the parameter β of a Leonard algebra L is such that q $(q + q^{-1} = \beta)$ is not a primitive nth root of unity for any n $(n \geq 3)$. Then $\sigma(a), \sigma(a^*)$ is a thin Leonard pair for any finite dimensional, irreducible, symmetric representation σ of L.

Proof of (i). Let $A, A^* \in End_{\mathbb{R}}(V)$ be a thin Leonard pair, with

$Spec(A) = (\theta_0, \theta_1, \ldots, \theta_d)$ and $Spec(A^*) = (\theta_0^*, \theta_1^*, \ldots, \theta_d^*)$

$(\theta_i, \theta_i^* \in \mathbb{R},\ 0 \leq i \leq d)$. A, A^* is irreducible by (1.3), (1.4), and V possesses the required bilinear form by (2.8), so it suffices to prove $a = A$, $a^* = A^*$ satisfy relations (3.1), (3.2) for some $\beta, \gamma, \gamma^*, \delta, \delta^*, \omega \in \mathbb{R}$. This is trivial if $d = 0$, so assume $d \geq 1$. Then if $d \geq 3$, by Bannai and Ito [3,p.288] there exists an $\beta \in \mathbb{R}$ where

$$\begin{aligned} &\theta_{i+3} - (\beta+1)\theta_{i+2} + (\beta+1)\theta_{i+1} - \theta_i \\ &= \theta_{i+3}^* - (\beta+1)\theta_{i+2}^* + (\beta+1)\theta_{i+1}^* - \theta_i^* \\ &= 0 \qquad\qquad (0 \leq i \leq d-3). \end{aligned}$$

If $d = 1$ or 2 let $\beta \in \mathbb{R}$ be arbitrary. If $d \geq 2$, set

$$\gamma = \beta\theta_1 - \theta_0 - \theta_2, \qquad \gamma^* = \beta\theta_1^* - \theta_0^* - \theta_2^*, \tag{3.4}$$

and if $d = 1$ let $\gamma, \gamma^* \in \mathbb{R}$ be arbitrary. Now set

$$\delta = \beta\theta_0\theta_1 - \theta_0{}^2 - \theta_1{}^2 - \gamma(\theta_0 + \theta_1), \qquad (3.5)$$

$$\delta^* = \beta\theta_0{}^*\theta_1{}^* - \theta_0{}^{*2} - \theta_1{}^{*2} - \gamma^*(\theta_0{}^* + \theta_1)$$

and define polynomials

$$p(x,y) = x^2 - \beta xy + y^2 + \gamma(x + y) + \delta,$$

$$p^*(x,y) = x^2 - \beta xy + y^2 + \gamma^*(x + y) + \delta^*.$$

Then

$$p(\theta_i,\theta_{i+1}) = p^*(\theta_i{}^*,\theta_{i+1}{}^*)$$

$$= 0 \qquad\qquad (0 \leq i \leq d-1). \qquad (3.6)$$

For p this follows from (3.5) if $i = 0$, (3.4) if $i = 1$, and

$$(p(\theta_i,\theta_{i+1}) - p(\theta_{i-1},\theta_i))(\theta_{i+1} - \theta_{i-1})^{-1}$$

$$- (p(\theta_{i-1},\theta_i) - p(\theta_{i-2},\theta_{i-1}))(\theta_i - \theta_{i-2})^{-1}$$

$$= \theta_{i+1} - (\beta+1)\theta_i + (\beta+1)\theta_{i-1} - \theta_{i-2} \qquad (3.7)$$

$$= 0$$

if $2 \leq i \leq d-1$. The case of p^* is similar. It now suffices to set

$$f_1 = A^2A^* - \beta AA^*A + A^*A^2 + \gamma(A^*A + AA^*) + \delta A^* + \gamma^*A^2 \qquad (3.8)$$

$$f_1{}^* = A^{*2}A - \beta A^*AA^* + AA^{*2} + \gamma^*(AA^* + A^*A) + \delta^*A + \gamma A^{*2} \qquad (3.9)$$

and prove $f_1 + \omega A = z\mathbb{I}$, $f_1{}^* + \omega A^* = z^*\mathbb{I}$ for some $\omega, z, z^* \in \mathbb{R}$. We first note $f_1 V_i \subset V_i$ $(0 \leq i \leq d)$, for if $i \neq j$ $(0 \leq i, j \leq d)$ then $E_i f_1 E_j = E_i A^* E_j p(\theta_i, \theta_j)$ vanishes if $|i - j| \geq 2$ by (1.2), and if $|i - j| = 1$ by (3.6). In particular f_1 commutes with A. Secondly,

$$f_1 V_0^* \subseteq V_0^* + V_1^*, \tag{3.10}$$

for if $d \geq 2$ then $f_1 V_0^* \subseteq V_0^* + V_1^* + V_2^*$ by (1.1), (3.8), with $E_2^* f_1 E_0^* = E_2^* A^2 E_0^* (\theta_0^* - \beta\theta_1^* + \theta_2^* + \gamma^*)$ vanishing by (3.4). Since $AV_0^* \subseteq V_0^* + V_1^*$, (3.10) implies some nonzero linear transformation $f_2 \in \text{Span}\{\mathbb{I}, A, f_1\}$ satisfies $f_2 V_0^* = 0$. But then f_2 commutes with A, so is identically 0 by (ii) of Lemma 2.1. We conclude $f_1 + \omega A = z\mathbb{I}$ for some $\omega, z \in \mathbb{R}$, and similarily $f_1^* + \omega^* A^* = z^* \mathbb{I}$ for some $\omega^*, z^* \in \mathbb{R}$. But then $\omega = \omega^*$, for (3.8), (3.9) immediately give

$$\begin{aligned} \omega^*(AA^* - A^*A) &= f_1^* A - A f_1^* \\ &= A^* f_1 - f_1 A^* \\ &= \omega(AA^* - A^*A). \end{aligned}$$

<u>Proof of (ii)</u>. let $p(x, y)$ be the polynomial $x^2 - \beta xy + y^2 + \gamma(x + y) + \delta$, where β, γ, δ are parameters of L. Let f be the expression in (3.1), and let $\sigma: L \to \text{End}_{\mathbb{R}}(V)$ be self-adjoint with respect to some positive definite form $\langle\, ,\rangle$ on V. Now f is symmetric in a, a^*, so $\sigma(f)$ is self-adjoint with respect to $\langle\, ,\rangle$. In particular it has an eigenvalue $z \in \mathbb{R}$. Now $\sigma(f) = z\mathbb{I}$ by irreducibility. Since $A = \sigma(a)$ is self-adjoint with respect

to $\langle , \rangle$ it is diagonalizable, with maximal eigenspaces denoted by $V_0, V_1, \ldots, V_d$ for some integer d $(d \geq 0)$. Denote by $\theta_0, \theta_1, \ldots, \theta_d$ and $E_0, E_1, \ldots, E_d$ respectively, the corresponding eigenvalues and projections. Simplifying $E_i \sigma(f) E_j$ $(0 \leq i,j \leq d)$, we find $A^* = \sigma(a^*)$ satisfies

$$p(\theta_i, \theta_j) E_i A^* E_j = 0 \qquad (i \neq j,\ 0 \leq i,j \leq d), \tag{3.11}$$

$$p(\theta_i, \theta_i) E_i A^* E_i = (z - \omega\theta_i - \gamma^*\theta_i^2) E_i \qquad (0 \leq i \leq d). \tag{3.12}$$

Also $E_i A^* E_j \neq 0$ implies $E_j A^* E_i \neq 0$ $(0 \leq i,j \leq d)$ by the self-adjointness of A^*. By irreducibility, and since p is symmetric and quadratic in (3.11), for an appropriate ordering of the V_i's we have

$$\{ (i, j) \mid E_i A^* E_j \neq 0,\ 0 \leq i < j \leq d \} \text{ equals}$$

$$\{ (i,i+1) \mid 0 \leq i \leq d-1 \} \quad \text{or} \quad \{ (i,i+1) \mid 0 \leq i \leq d-1 \} \cup \{ (0,d) \}.$$

But the second possibility cannot occur, for $p(\theta_0, \theta_1) = p(\theta_1, \theta_2) = \ldots = p(\theta_{d-1}, \theta_d) = p(\theta_d, \theta_0) = 0$, (3.7), and Lemma 3.2 (with $n = d+1$, $r_i = \theta_i (0 \leq i \leq d)$) contradict our assumption concerning q. Interchanging the roles of A, A^*, we may also assume

$$E_i^* A E_j^* \begin{cases} \neq 0 & \text{if } |i-j| = 1 \\ = 0 & \text{if } |i-j| \geq 2 \end{cases} \qquad (0 \leq i,j \leq e),$$

where $E_0^*, E_1^*, \ldots, E_e^*$ are the projections onto the maximal eigenspaces $V_0^*, V_1^*, \ldots, V_e^*$ of A^*. Now we are done if we can show $\dim(V_i) = \dim(V_j^*) = 1$ $(0 \leq i \leq d,\ 0 \leq j \leq e)$. This is trivial if d or e is 0, so assume $d, e \geq 1$. Let $v_0 \in V_0^*$ be an eigenvector for the self adjoint map $E_0^*AE_0^*$, with associated eigenvalue $\lambda \in \mathbb{R}$, and set $v_i^* = E_i v_0$ $(0 \leq i \leq d)$. Also set $v_1 = (A-\lambda I)v_0 \in V_1^*$. To prove $\dim(V_i) = 1$ $(0 \leq i \leq d)$, it suffices to show $W = \mathrm{Span}\{v_0^*, v_1^*, \ldots, v_d^*\}$ is A- and A^*- invariant, and hence equal V. A-invariance is immediate, so we are done if

$$E_iA^*v_{i-1}^*, \quad E_iA^*v_i^*, \quad E_iA^*v_{i+1}^* \in W \qquad (0 \leq i \leq d). \qquad (3.13)$$

But $v_0^* + v_1^* + \ldots + v_d^* = v_0$, $\quad A^*v_0 = \theta_0^* v_0$ give

$$E_iA^*v_{i-1}^* + E_iA^*v_i^* + E_iA^*v_{i+1}^* = \theta_0^* v_i^* \qquad (0 \leq i \leq d), \qquad (3.14)$$

and $(\theta_0-\lambda)v_0^* + (\theta_1-\lambda)v_1^* + \ldots + (\theta_d-\lambda)v_d^* = v_1$, $A^*v_1 = \theta_1^* v_1$ give

$$(\theta_{i-1}-\lambda)E_iA^*v_{i-1}^* + (\theta_i-\lambda)E_iA^*v_i^* + (\theta_{i+1}-\lambda)E_iA^*v_{i+1}^*$$

$$= \theta_1^*(\theta_i-\lambda)v_i^* \qquad (0 \leq i \leq d), \qquad (3.15)$$

so at least (3.13) holds for $i = 0, d.$, Now $p(\theta_i, \theta_i) \neq 0$ $(1 \leq i \leq d-1)$, for otherwise the nonzero quadratic polynomial $f(x) = p(x, \theta_i)$ has roots $\theta_{i-1}, \theta_i, \theta_{i+1}$, a contradiction. Therefore $E_iA^*v_i^* \in W$ $(1 \leq i \leq d-1)$ by (3.12), a fact which we may combine with (3.14), (3.15) to obtain (3.13) for the cases $1 \leq i \leq d-1$. This shows $\dim(V_i) = 1$ $(0 \leq i \leq d)$. Showing $\dim(V_j^*) = 1$ $(0 \leq j \leq e)$ is similar. This completes the proof of (ii) and the Theorem. □

3. QUASI-BIPARTITE LEONARD PAIRS

In this section we classify the symmetric quasi A- or A*-bipartite Leonard pairs. Without loss we assume quasi A*-bipartite throughout. We first look at the irreducible case.

THEOREM 4.1. Let $A, A^* \in \mathrm{End}_{\mathbb{R}}(V)$ be an irreducible, symmetric, quasi A*-bipartite Leonard pair. Then A, A^* is thin.

<u>Proof</u>. We must show $\dim(V_i) = \dim(V_j^*) = 1$ $(0 \leqslant i \leqslant d,\ 0 \leqslant j \leqslant e)$ for the maximal eigenspaces $V_0, V_1, \ldots, V_d$ of A and $V_0^*, V_1^*, \ldots, V_e^*$ of A^*. The argument for the V_i's is the same as the corresponding argument at the end of the proof of Theorem 3.4 (ii), except that now $E_iA^*v_i^*=0$ $(1 \leqslant i \leqslant d-1)$ in the paragraph following (3.15). Now if $v \in V_0$ $(v \neq 0)$, then $v, A^*v, A^{*2}v, \ldots, A^{*d}v$ are independent by (1.3), making the degree $e+1$ of the minimal polynomial of A^* at least $d+1$. But then $d+1 \leqslant e+1 \leqslant \dim(V_0^*) + \ldots + \dim(V_e^*) = \dim(V) = d+1$ forces $\dim(V_i^*) = 1$ $(0 \leqslant i \leqslant e)$. This proves A, A^* is thin. □

To extend the above result we need the following notation. For any Leonard pair A, A^* in Definition 1.1, denote by $S = S(A,A^*)$ (resp. $S^* = S^*(A,A^*)$) the set of eigenvalues θ_i of A for which $E_iA^*E_i \neq 0$ $(0 \leqslant i \leqslant d)$ (resp. the eigenvalues θ_i^* of A^* for which $E_i^*AE_i^* \neq 0$ $(0 \leqslant i \leqslant e)$). Recall the <u>direct sum</u> of real vector spaces V, W is the vector space $V \oplus W = \{(v,w) \mid v \in V, w \in W\}$. Also, if $C \in \mathrm{End}_{\mathbb{R}}(V)$ and $D \in \mathrm{End}_{\mathbb{R}}(W)$

then $C\oplus D \in End_{\mathbb{R}}(V\oplus W)$ is the map defined by $C\oplus D(v,w) = (Cv, Dw)$ $(v \in V, w \in W)$. A sequence $r = \{r_0, r_1, \ldots, r_n\}$ is a <u>subsequence</u> of a sequence $s = \{s_0, s_1, \ldots, s_m\}$ if $r_i = s_{i+t}$ $(0 \leq i \leq n)$ for some integer t $(0 \leq t \leq m-n)$. We now have

THEOREM 4.2. Let $\underline{\theta} = (\theta_0, \theta_1, \ldots, \theta_d)$, $\underline{\theta}^* = (\theta_0^*, \theta_1^*, \ldots, \theta_e^*)$ be any finite sequences, each with distinct real entries. Let $A_1, A_1^* \in End_{\mathbb{R}}(V_1)$, $A_2, A_2^* \in End_{\mathbb{R}}(V_2), \ldots, A_n, A_n^* \in End_{\mathbb{R}}(V_n)$ be a finite collection of thin, symmetric Leonard pairs, such that

(1) $Spec(A_i)$ is a subsequence of $\underline{\theta}$ $(1 \leq i \leq n)$

(2) $Spec(A_i^*)$ is a subsequence of $\underline{\theta}^*$ $(1 \leq i \leq n)$

(3) $S(A_i, A_i^*) \subseteq (\theta_0, \theta_d)$ $(1 \leq i \leq n)$

(4) for each integer i $(0 \leq i \leq d-1)$, there exists an integer j $(1 \leq j \leq n)$ where (θ_i, θ_{i+1}) is a subsequence of $Spec(A_j)$ and

(5) for each integer h $(0 \leq h \leq e-1)$, there exists an integer k $(1 \leq k \leq n)$ where $(\theta_h^*, \theta_{h+1}^*)$ is a subsequence of $Spec(A_k^*)$.

Then $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$, $A^* = A_1^* \oplus A_2^* \oplus \ldots \oplus A_n^* \in End_{\mathbb{R}}(V_1 \oplus V_2 \oplus \ldots \oplus V_n)$ is a symmetric, quasi A^*- bipartite Leonard pair with $Spec(A) = \underline{\theta}$, $Spec(A^*) = \underline{\theta}^*$. Furthermore, any symmetric quasi A^*- bipartite Leonard pair is equivalent to a Leonard pair of this form.

Proof. That our construction yields the required Leonard pair is immediate, so consider our last assertion. Let $A, A^* \in End_{\mathbb{R}}(V)$ be symmetric and quasi A^*- bipartite. Then A and A^* are each self-adjoint with respect to some positive definite bilinear form $\langle\, ,\, \rangle$ on V. Now if a subspace $Y \subseteq V$ is A- and A^*- invariant, then so is $Y^{\perp} = \{ v \mid \langle v, y\rangle = 0 \text{ for all } y \in Y\}$, so by induction $V = V_1 \oplus V_2 \oplus \ldots \oplus V_n$ (orthogonal direct sum), with each V_i nonzero and minimal with respect to being A- and A^* - invariant. Now each pair $A|_{V_i}, A^*|_{V_i^*}$ $(1 \leq i \leq n)$ is an irreducible, symmetric Leonard pair, and therefore thin by Theorem 4.1. Now (1) - (3) above are immediate, with (4), (5) following from (1.3), (1.4). □

We mention some open problems.

1. Find all irreducible representations σ of the Leonard algebra in Theorem 3.4 (ii), with nothing assumed about q or the self- adjointness of σ. The representation theory of Leonard algebras over fields other than $\mathbb{R}$ may also be of interest.

2. Let $A, A^* \in End_{\mathbb{R}}(V)$ be an irreducible Leonard pair of diameters d, e. We conjecture that there exist integers b, c $(0 \leq b \leq d,\ 0 \leq c \leq e)$ where

$$1 = \dim(V_0) \leq \dim(V_1) \leq \ldots \leq \dim(V_{b-1}) \leq \dim(V_b)$$

$$\dim(V_b) \geq \dim(V_{b+1}) \geq \ldots \geq \dim(V_d) = 1$$

and

$$1 = \dim(V_0^*) \leq \dim(V_1^*) \leq \ldots \leq \dim(V_{c-1}^*) \leq \dim(V_c^*)$$

$$\dim(V_c^*) \geq \dim(V_{c+1}^*) \geq \ldots \geq \dim(V_e^*) = 1.$$

3. We may generalize the notion of a thin Leonard pair as follows. Let $A, A^* \in End_{\mathbb{R}}(V)$ be diagonalizable, with one dimensional maximal eigenspaces $V_0, V_1, \ldots, V_d$ and $V_0^*, V_1^*, \ldots, V_d^*$, respectively. Assume V has no proper nonzero A- and A*- invariant subspaces. Now define a diagram D (resp. D*) on the nodes 0, 1, ..., d by drawing a directed arc from any node i to any node j for which $E_i A^* E_j \neq 0$ (resp. $E_i^* A E_j^* \neq 0$). Here E_i, E_i^* are the projections onto the eigenspaces. Then D, D* are strongly connected by the irreducibility of V. If there exists a positive definite bilinear form with respect to which A, A* are self-adjoint, the diagrams are essentially undirected. What pairs D, D* can appear? What are the families of D, D* for which there is an analog to Leonard's theorem?

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